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Department of Computer Science, National Tsing Hua University
Qualification Examination, Algorithms, Fall 2016

1. (8%) Let $f(n)$, $g(n)$, and $h(n)$ be asymptotically nonnegative functions such that $f(n) = O(g(n) - 200)$ and $g(n) = O(h(n) / 9)$. Using the definition of O-notation, prove or disprove that $f(n) = O(h(n))$.

2. (10%) Find an upper bound on recurrence $T(n) = T(n - 5) + T(5) + n$ by using the substitution method. Assume that $T(n) = 1$ for $1 \le n \le 5$.

3. (10%)
   (a) (6%) Describe a divide-and-conquer algorithm for the 2-dimensional closest pair problem. The time complexity of your algorithm should be better than $O(n^2)$.
   (b) (4%) What is the time complexity of your algorithm? Explanations are necessary.

4. (10%) A *coloring* is an assignment of colors to each node of a graph such that adjacent nodes have different colors. The *k-coloring problem* is to determine whether a given graph can be colored by using at most k colors. It is known that the 3-coloring problem is NP-complete. Prove or disprove that the 4-coloring problem is also NP-complete.

5. (7%) Please give an $O(|V||E|)$-time algorithm for computing the transitive closure of a directed graph $G = (V, E)$.

6. (10%) Please give a 2-approximation algorithm for the vertex-cover problem. Analysis of the approximation ratio is necessary.

7. (12%)
   (a) (8%) Please give a linear-time algorithm to determine whether a text T_1 is a substring of another string T_2.
   (b) (4%) Please give a linear-time algorithm to determine whether a text T_1 is a cyclic rotation of another string T_2. For example, "arc" and "car" are cyclic rotations of each other. [You can apply the result of (a) even if you cannot answer (a).]

8. (10%) Given an array $A[1..k]$ of k strings, with each string representing a distinct positive integer. The characters in each string are chosen from ['0', '9'], and each string does not start with '0'. For instance, $A[i]$ may store "1203", which has four characters and represents the integer 1203.

   Let n denote the total number of characters in the strings. Show how to sort the strings in $O(n)$

time in alphabetical order.

[No marks if your algorithm does not run in $O(n)$ time, or if it is incorrect.]

9. (10%) Let $B = (b_1, b_2, \ldots, b_n)$ be a sequence of positive integers. Design an $O(n^2)$ algorithm that finds a strictly increasing subsequence, that is $(b_{i_1}, b_{i_2}, \ldots, b_{i_j})$ with $i_1 < i_2 < \ldots < i_j$ and $b_{i_1} < b_{i_2} < \ldots < b_{i_j}$, such that the sum of all terms in the subsequence is maximized.

   [No marks if your algorithm does not run in $O(n^2)$ time, or if it is incorrect.]

10. (13%) A directed graph is said to be *semi-connected* if for any pair of distinct vertices u and v, there exists either a directed path from u to v, or a directed path from v to u, or both.

    (a) (5%) Let G be an acyclic directed graph with n vertices. Suppose that we have performed a topological sort on G, and v_i denotes the ith vertex in the topological sort order.

    Show that G is semi-connected if and only if there is a directed edge (v_i, v_{i+1}) for every $i = 1, 2, \ldots, n-1$.

    (b) (8%) Let $H = (V, E)$ be a general directed graph (which may contain a cycle). Give an $O(|V| + |E|)$-time algorithm that determines if H is semi-connected or not. Explain why your algorithm is correct.

    [No marks if your algorithm does not run in $O(|V| + |E|)$ time, or if the correctness is not explained clearly.]

    Hint: Compute strongly connected components, and apply the result of (a).

    Note: Part (a) and part (b) will be scored independently. You may assume that the result of part (a) is correct when writing your solution for part (b).