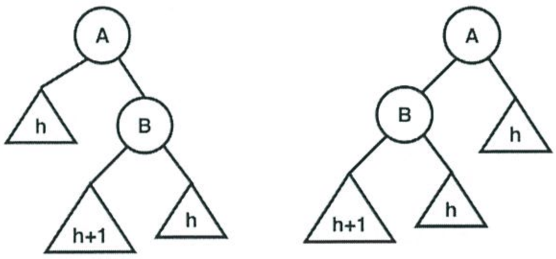
# Department of Computer Science, National Tsing Hua University
## Qualification Examination, Algorithms, Fall 2017

1. (10%) Answer the following questions:
   (a) What is the maximum number of nodes in a binary tree of height $h$?
   (b) The worst-case time complexity of finding the node with minimum key value in a min-heap of size $N$ is $\Theta(\log N)$. True or False?
   (c) Sort the following time complexity expressions in ascending order:
   $$O(n^2);\quad O(n);\quad O(n\log n);\quad O(\log n);\quad O(2^n);$$
   (d) The worst-case complexity of quick sort algorithm is $\Theta(n\log n)$. True or False?
   (e) How many undirected simple graphs (not necessarily connected) can be constructed out of a given set $V$ of $n$ vertices?

2. (10%) Suppose that the output of an *inorder* traversal on a max-heap $H$ is
   2, 16, 7, 62, 5, 9, 188, 14, 78, 10.
   Reconstruct and draw the max heap $H$.

3. (15%) AVL tree is a self-balancing binary search tree (BST), which is capable of fixing the unbalanced sub-trees caused by BST insert/delete operations.
   (a) Illustrate how AVL tree handles the following two unbalanced sub-trees.




   (b) Draw the sequence of AVL trees formed by inserting the integer keys 9, 27, 50, 15, 2, 21, and 36 successively into an initially empty AVL tree.

4. (15%) Let $I$ denote a collection of $n$ intervals, with distinct integral endpoints chosen from $[1, n^2]$. Each interval in $I$ can be expressed by $[l, r]$, with $l < r$, indicating its endpoints. We say two intervals $[l_1, r_1]$ and $[l_2, r_2]$ are *disjoint* if $r_1 < l_2$ or $r_2 < l_1$; otherwise, the intervals are *overlapping*.
   Give an $O(n)$ time algorithm to count the number of pairs of intervals in $I$ that are overlapping.
   *Example:* Suppose $I$ has three intervals $x = [1, 3]$, $y = [2, 5]$, $z = [4, 6]$. Then, $x$ and $y$ are overlapping, $y$ and $z$ are overlapping, but $x$ and $z$ are not, so that the desired count is 2.

5. (10%) Let $G = (V, E)$ be an undirected graph with non-negative edge weights. Give the time complexity of Warshall-Floyd algorithm in terms of $|V|$ and $|E|$, and explain how it solves the all-pairs shortest paths problem on $G$.

6. (10%) There are stones with a total mass of 9 tons that should be transported by trucks. Stones may have different masses, but none of the stones is heavier than 1 ton. Each truck has a capacity of 3 tons, and to transport each stone, we must put the stone as a whole on one truck (i.e., we cannot break a stone into smaller pieces and deliver by different trucks).

(a) (5%) Show that there exists a scenario where we need at least 4 trucks to transport all the stones at the same time.

(b) (5%) Consider allocating the stones by the AnyFit algorithm as follows:

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Start with an empty truck T1. Set i = 1.
while (there is a stone s not in any truck) {
  if (any of the trucks T1, T2, . . . , Ti can hold s)
     Put s in some truck that can hold s;
  else
     Put s in a new empty truck Ti+1, and then update i as i + 1.
}
```

Show that by running AnyFit, at most 4 trucks are used in any scenario.

7. (15%) Consider the following matrix multiplication

$$\begin{bmatrix} f_0 \\ f_1 \end{bmatrix} = \begin{bmatrix} d_0 & d_1 & d_2 \\ d_1 & d_2 & d_3 \end{bmatrix} \begin{bmatrix} g_0 \\ g_1 \\ g_2 \end{bmatrix}.$$

Suppose that you do not have the access to the values of $d_i$ and $g_j$ for any $i$ or $j$. Instead, you are given the values $m_1, m_2, m_3, m_4$ which are defined as follows:

$$m_1 = (d_0 - d_2)g_0,$$
$$m_2 = (d_1 + d_2)(g_0 + g_1 + g_2)/2,$$
$$m_3 = (d_2 - d_1)(g_0 - g_1 + g_2)/2$$
$$m_4 = (d_1 - d_3)g_2.$$

Show how to use $m_1, m_2, m_3, m_4$ to compute $f_0$ and $f_1$.

8. (10%) Write a recursive algorithm to compute the greatest common divisor (GCD) of two non-negative numbers, $a$ and $b$. You may assume that GCD of 0 and 0 is 0.

9. (5%) Explain what a polynomial-time approximation scheme (PTAS) is.